

0.8 Euler Method

Consider the IVP

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \quad (1)$$

$t_0 = a$, $t_N = b$ and $t_{i+1} = t_i + h$, $i = 0, 1, \dots, N-1$.

Euler Method:

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf(t_i, w_i), \quad i = 0, 1, \dots, N-1$$

0.8.1 Theorem 5.9(book): Bound for Global Error

- f continuous and satisfies Lipschitz condition on $D = \{(t, y) | t \in [a, b], y \in (-\infty, \infty)\}$.
 - There is M such that $|y''(t)| \leq M$, for all $t \in [a, b]$
 - $y(t)$ unique solution for (1).
 - $w_0, w_1, \dots, w_N$ the approximations of Euler Method.
- Then,

$$|y(t_i) - w_i| \leq \frac{hM}{2L} [e^{L(t_i-a)} - 1] \quad (2)$$

1 Section 5.10: Stability and Convergence

Consider the IVP

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \quad (3)$$

and the one-step finite difference method (FDM)

$$w_0 = \alpha, \quad w_{i+1} = w_i + h\phi(t_i, w_i, h), \quad i = 0, \dots, N-1 \quad (4)$$

with local truncation error (LTE) at the i th step $\tau_i(h)$.

1.1 Definition: Consistency

The FDM (4) is consistent with the IVP (3) if

$$\lim_{h \rightarrow 0} \|\tau(h)\|_{\infty} = \lim_{h \rightarrow 0} \max_{1 \leq i \leq N} |\tau_i(h)| \rightarrow 0,$$

$$\text{where } \tau(h) = \begin{pmatrix} \tau_1(h) \\ \tau_2(h) \\ \vdots \\ \tau_N(h) \end{pmatrix}$$

1.2 Definition: Convergence

The FDM (4) is convergent with respect to the IVP (3) if

$$\lim_{h \rightarrow 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| \rightarrow 0$$

1.3 Definition: Stability

The FDM (4) is stable if given two solutions $\{u_i\}_0^N$ and $\{v_i\}_0^N$ such that their initial values satisfy $|u_0 - v_0| < \epsilon$, then there exists $K > 0$ such that

$$|u_i - v_i| < K\epsilon, \quad 1 \leq i \leq N$$

It means that small changes in the initial conditions produce correspondingly small changes in the subsequent approximations. Compare with well-posedness of an IVP.

1.4 Theorem 5.20: Stability-Convergence-Consistency

If there is $h_0 > 0$ such that $\phi(t, w, h)$ of FDM (4) is continuous and satisfies a Lipschitz condition in w with constant L on the set

$$D = \{(t, w, h) \mid t \in [a, b], w \in (-\infty, \infty), 0 \leq h \leq h_0\}$$

Then,

1. The method is stable.
2. The FDM is convergent if and only if it is consistent. This is equivalent to

$$\phi(t, y, 0) = f(t, y), \quad a \leq t \leq b$$

3. If a function $\eta(h)$ exists such that $|\tau_i(h)| \leq \eta(h)$ when $0 \leq h \leq h_0$, then

$$|y(t_i) - w_i| \leq \frac{\eta(h)}{L} e^{L(t_i - a)}$$

1.5 Consistency and Convergence of Multistep Methods

Consider a general m -step multistep method

$$\begin{aligned} w_0 &= \alpha, \quad w_1 = \alpha_1, \dots, \quad w_{m-1} = \alpha_{m-1}, \\ w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \dots + a_0w_{i-(m-1)} + hF(t, h, w_{i+1}, w_i, \dots, w_{i-(m-1)}), \end{aligned} \quad (5)$$

for $i = m-1, m, \dots, N-1$.

1.6 Definition: Local Truncation error

If $y(t_i) = y_i$, for $i = 0, \dots, N$ are the values of the exact solution evaluated at the grid points, then the local truncation error is defined as

$$\tau_{i+1}(h) = \frac{y_{i+1} - a_{m-1}y_i - \dots - a_0y_{i-(m-1)}}{h} - F(t, h, y_{i+1}, y_i \dots y_{i-(m-1)}), \quad (6)$$

for $i = m - 1, m, \dots, N - 1$.

1.7 Definition: Lipschitz Condition for Multistep Methods

The function F in the definition of multistep methods (5) satisfies a Lipschitz condition with respect to the sequence $\{w_j\}_0^N$, if there is $L > 0$ such that for every pair of sequences $\{v_j\}_0^N, \{\tilde{v}_j\}_0^N$

$$|F(t_i, h, v_{i+1}, \dots, v_{i-(m-1)}) - F(t_i, h, \tilde{v}_{i+1}, \dots, \tilde{v}_{i-(m-1)})| \leq L \sum_{j=0}^m |v_{i+1-j} - \tilde{v}_{i+1-j}| \quad (7)$$

for $i = m - 1, \dots, N - 1$.

1.8 Definition: Consistency for Multistep Methods

The multistep method (5) is consistent if

$$\lim_{h \rightarrow 0} |\tau_i(h)| = 0, \quad i = m, \dots, N$$

and

$$\lim_{h \rightarrow 0} |\alpha_i - y_i| = 0, \quad i = 1, \dots, m - 1.$$

This last condition may be interpreted as requiring convergence of the “starting method”.

1.9 Theorem 5.21: Local Truncation Error, Consistency and Convergence for Adams Multistep Methods

Consider the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

which is approximated by an m -step explicit predictor-corrector Adams's method. If

1. the predictor part has local truncation error $\tau_{i+1}(h)$ and the corrector part has LTE $\tilde{\tau}_{i+1}(h)$,
2. f and $\frac{\partial f}{\partial y}$ are continuous on $D = \{(t, y) | t \in [a, b], y \in (-\infty, \infty)\}$, and $\frac{\partial f}{\partial y}$ is bounded on D . Therefore, f satisfies a Lipschitz condition on D .

Then,

1. the local truncation error of the predictor-corrector is given by

$$\sigma_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \tau_{i+1}(h) \tilde{b}_{m-1} \frac{\partial f}{\partial y}(t_{i+1}, \theta_{i+1}), \quad (8)$$

where θ_{i+1} is between 0 and $h\tau_{i+1}(h)$.

2. There are constants k_1 and k_2 such that

$$|w_i - y(t_i)| \leq \left[\max_{0 \leq j \leq m-1} |w_j - y_j| + k_1 \sigma(h) \right] e^{k_2(t_i - a)}, \quad (9)$$

where $\sigma(h) = \max_{m \leq j \leq N} |\sigma_j(h)|$

Discuss this theorem.

1.10 Definition: Characteristic Polynomial

The characteristic polynomial corresponding to (5) is defined as

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \dots - a_1\lambda - a_0$$

$$w_{i+1} - a_{m-1} w_i - a_{m-2} w_{i-1} + \dots + a_1 w_{i-(m-2)} + a_0 w_{i-(m-1)}$$

*So, it is consistent
and $\tau_{i+1}(h) = O(\min(\sigma(\tau_{i+1}(h)), \tilde{\tau}_{i+1}(h)))$
So, it is convergent
And order of convergence depends
on both the initial method and
the P-C method.*

1.11 Definition: Root Condition

If $\lambda_1, \dots, \lambda_m$ are the roots of the characteristic polynomial

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \dots - a_1\lambda - a_0$$

corresponding to the multistep method (5),

- i) $|\lambda_i| \leq 1 \quad i = 1, \dots, m$ and
- ii) all roots with absolute value 1 are simple roots.

Then, the difference method is said to satisfy a **root condition**.

Discuss this concept.

1.12 Definition: Stability

1. Multistep methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation of magnitude 1 are called strongly stable.
2. Multistep methods that satisfy the root condition and have more than one distinct root with magnitude 1 are called weakly stable.
3. Multistep methods that do not satisfy the root condition are called unstable.

1.13 Theorem 5.24: Stability, Consistency, and Convergence of General Multistep Methods

If the m -step multistep method

$$\begin{aligned} w_0 &= \alpha, \quad w_1 = \alpha_1, \dots, \quad w_{m-1} = \alpha_{m-1}, \\ w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \dots + a_0w_{i-(m-1)} + hF(t, h, w_{i+1}, w_i, \dots, w_{i-(m-1)}), \end{aligned} \quad (10)$$

for $i = m-1, m, \dots, N-1$, is consistent then,
it is stable if and only if it is convergent.

Discuss it.